You Could Have Discovered the Signature Theorem

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This is a continuation and completion of the talks I gave on the signature theorem. The goal is to not just prove the theorem, which we have done by checking the two functions on basis, but show how one could arrive at the conjecture in the first place. We will follow [hir71].

The question we want to answer is: which power series is the signature? From [MS16] we have the following facts:

1. It would be sufficient to answer, which multiplicative sequence is the signature as we can obtain the power series by evaluating on $1 + t$.
2. The signature of complex projective spaces is 1, and it is sufficient for the multiplicative sequence to agree on complex projective spaces.
3. Evaluating a multiplicative sequence is given by
$$K\left(\mathbb{C}P^{2n}\right) \quad := \quad K_n(p_1, ..., p_n)[\mathbb{C}p^{2n}]$$
where the p_i denote the Pontryagin classes and square brackets the fundamental class.
4. Given a generator of $\alpha \in H^2(\mathbb{C}P^n)$ we have the total Pontryagin class
$$p(\mathbb{C}P^n) = 1 + \sum_i p_i = (1 + \alpha^2)^{2n+1}$$
5. The sequence will be multiplicative in the total class.

4 and 5 show that

$$K(p(\mathbb{C}P^{2n})) = K((1 + \alpha^2)^{2n+1}) = K(1 + \alpha^2)^{2n+1}, \quad \forall n$$

Because the Pontryagin classes are all polynomials in a single cohomology class for $\mathbb{C}P^{2n}$ we actually know that K_n is a homogeneous polynomial of degree n in α^2, thus a monomial of degree n in α^2. Thus 2 and 3 simplifies to

$$K\left(\mathbb{C}P^{2n}\right) = \lambda_n \alpha^{2n}[\mathbb{C}p^{2n}] = \lambda_n = 1$$

or in other words the coefficient of α^{2n} is 1, *in the power series for* $\mathbb{C}P^{2n}$. Now combining $1, 5$ we see that

$$K(p(\mathbb{C}P^{2n})) = 1 + \sum_i K_i(p_1, ..., p_i) = 1 + \sum_i \lambda_i \alpha^{2i} = K(1 + \alpha^2)^{n+1} = f(\alpha^2)^{2n+1}.$$

We need to carefully note that these power series are being evaluated on the class of $\mathbb{C}P^{2n}$ and are therefore truncated. Putting it together we require a power series such that for every n the coefficient of x^n in $f(x)^{2n+1}$ is 1.

Remark. This is not the same as requiring that

$$f(t)^{2n+1} = \sum_{i\geq 0} t^i$$

In particular the coefficient of $x^{n\pm 1}$ say *might not be 1* in $f(x)^{2n+1}$.

Lemma. *The only power series such that for every n the coefficient of x^n in $f(x)^{n+1}$ is one is given by*

$$f(x) = \frac{x}{1-e^{-x}} = 1 + \frac{x}{2} + \sum_{k\geq 1}(-1)^{k+1}B_k\frac{x^{2k}}{(2k)!}$$

Remark: This isnt the same relation however it is similar (it is the relation for the Todd genus). We *probably* want a power series in x^2 so that when we "square root" this power series we still get a power series in t (something like this is what happens in the Milnor-Stasheff proof).

Proof: We want the coefficient of z^n in the meromorphic function

$$\left(\frac{z}{1-e^{-z}}\right)^{n+1}$$

by Lauraunts theorem this is given by

$$\frac{1}{2\pi i}\oint\left(\frac{z}{1-e^{-z}}\right)^{n+1}\frac{1}{z^{n+1}}dz = \text{Res}_{z=0}\frac{1}{(1-e^{-z})^{n+1}} = 1$$

Calculating this residue is a bit annoying.

□

Now we see that $f(x) - \frac{x}{2}$ *is* a powerseries in x^2 it is also clear that

$$f(2x) - x = \frac{x}{\tanh x}$$

At this point we can conjecture that $x/\tanh x$ is involved and then prove the theorem as in Milnor-Stasheff, the exact process Hirzebruch followed is not clear in [hir71] however this is hinted at, that after some playing around with these series the one in the theorem was found.

Can we tie it all together?

References

[hir71] The Signature Theorem: Reminiscences and Recreation. In *Prospects in Mathematics. (AM-70), Volume 70*. Princeton University Press, 1971.

[MS16] John Willard Milnor and James D. Stasheff. *Characteristic Classes: AM-76*. Number 76 in Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2016.